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BMS field theory and holography in asymptotically flat space-times

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Abstract: We explore the holographic principle in the context of asymptotically flat space-times by means of the asymptotic symmetry group of this class of space-times, the so called Bondi-Metzner-Sachs (BMS) group. In particular we construct a (free) field theory living at future (or past) null infinity invariant under the action of the BMS group. Eventually we analyse the quantum aspects of this theory and we explore how to relate the correlation functions in the boundary and in the bulk.

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1. Introduction

Since 't Hooft foundational paper [1], the holographic principle played a key role in improving our understanding on the nature of gravitational degrees of freedom in a quantum field theory over a curved background. Originally the principle was proposed in order to solve the apparent information paradox of black holes by means of a theory living on a lower dimensional hypersurface (usually the boundary) with respect to bulk space-time where all the physical information of the manifold is encoded. Moreover, motivated by the Bekenstein entropy formula, the density of data on the "holographic screen" should not exceed the Planck density which implies that there is a high redundancy in the way we usually count degrees of freedom in a quantum field theory since if we excite more than $\frac{A}{4}$ degrees of freedom, we end up with a black hole.

A way to explicitly realize the holographic principle is based on the reconstruction of the bulk starting directly from boundary data explaining how they are generated, their dynamics and mainly how they can reproduce classical space-time geometry. An example is the AdS/CFT correspondence [2] (see [3] for a recent review) where, in the low energy limit, a supergravity theory living on $AdS_d \times M^{10-d}$ is a SU(N) conformal gauge field theory living on the boundary of AdS. The whole approach is based on the assumption of the equivalence of partition sum of gravity and gauge theory once asymptotically AdS boundary conditions are imposed on the bulk space-time. Thus it seems rather natural to investigate whether we could find a similar holographic description once we choose a different class of manifolds and thus of boundary conditions. In this paper we will address this problem for asymptotically flat space-times, a scenario where the quest for finding a suitable holographic description is extremely different from its AdS counterpart. Just to mention one, a key difference lies in the geometrical aspect of the hypersurface where the holographic data are encoded since, while in AdS the boundary is a lorentzian hypersurface, in an asymptotically flat manifold it is a null submanifold; this implies a greater difficulty than in AdS if one wishes to construct a theory living on the boundary.

Up to now different approaches have been proposed: in a recent one [4, 5], a Minkowski background is considered and it is divided in AdS and dS slices; the idea is to apply separately both AdS/CFT and dS/CFT correspondence and then patch together the results. This approach is interesting but it is limited up to now only to the flat manifold and it is unclear how to extend it to a generic asymptotically flat space-time.[1]

In this paper, instead, we continue along the road of [7, 8] where the holographic principle has been explored in asymptotically flat space-times by means of the asymptotic symmetry group at null infinity $\mathrm{Im}^{\pm}$, namely the Bondi-Metzner-Sachs group (BMS). Since in Penrose intrinsic construction of $\mathrm{Im}^{\pm}$, the BMS is the diffeomorphism group preserving the boundary metric, the underlying idea in the previous works was, besides pointing out the differences between the flat and the AdS scenario, to study the key ingredients of a field theory living on $\mathrm{Im}^{\pm}$ invariant under a diffeomorphism transformation. In particular in [7], by means of pure group theoretical techniques, we have shown the content of the full particle spectrum of a BMS field theory (richer than in a Poincaré invariant scenario), the possible wave functions and the associated equations of motion. Moreover, in [8], we have related the (unfaithful) representation of the BMS group with the infrared sectors of a pure gravitational system at null infinity and we have shown that it is possible to construct the dynamic of the BMS free particles namely their hamiltonians; the surprising result is the existence of a one to one correspondence between the (covariant) phase space of BMS and Poincaré massive $\mathrm{SU}(2)$ particle and between BMS $\mathrm{SO}(2)$ and Poincaré $E(2)$ particles. This implies that the boundary theory already at a classical level encodes all the information from the bulk (Poincaré invariant) theory; nonetheless this is not sufficient to claim about an holographic correspondence since the latter is fully manifest at a quantum level and we still lack a proper understanding of the quantum BMS field theory not to mention a way to reconstruct bulk data (correlation functions in particular) from boundary ones.

In this paper, where we adopt the language from the previous works, we address this problem starting from the free BMS fields; in particular we construct the BMS actions for all the possible wave functions and we study through path-integral techniques both the partition function and the 2-point correlation function for the leading example of a free massive (and massless) scalar field. Let us emphasize that we have chosen the case of a spinless particle only for sake of simplicity but, as we will clarify in the paper, the result

[1] Another recent paper [6] suggests to relate holography in a Ricci flat space-time to a Goursat problem i.e. a characteristic problem in a lorentzian setting.

could be easily obtained for any other field. Another natural question concerns the role of interactions both in the bulk and in the boundary; we will not directly address this problem in this paper since we believe that, before understanding the nature of a candidate holographic correspondence for free fields, the above question is premature. Nonetheless we wish to emphasize that the a key step in order to analyse the holographic counterpart of a bulk interaction would be to learn how interactions could be implemented in the boundary BMS field theory. At a classical level, the analysis performed in [8] suggests that we have to draw a distinction: if we consider a boundary particle such as a BMS scalar field ϕ interacting with a non trivial potential (a canonical example is $V(\phi)=\frac{\lambda}{4!}\phi^4$), the (covariant) phase space can be constructed as in the free field scenario and the 1:1 correspondence with the bulk counterpart should hold without modifications as it can be inferred from the analysis in the next section. Instead, a completely different scenario arises if we consider gauge interactions since we still lack a way to couple a BMS invariant field with a gauge potential; techniques such as the "minimal substitution", which have been made precise at a mathematical level in [9, 10] and [11], fail in a BMS setting since they strongly rely on symplectic deformation techniques proper of finite dimensional manifolds; thus the canonical overall process of substituting in an hamiltonian framework the momentum p^μ with $p^\mu - eA^\mu$ as in QED, cannot be blindly applied to the infinite dimensional calculus proper of a BMS field theory unless a procedure similar to the one proposed by Weinstein in [11] is developed within this setting. Thus the analysis of the role of bulk gauge fields is premature and, in this paper, we will not address further this point leaving it for a future analysis.

Instead, we will propose an answer to the question of the reconstruction of the bulk space-time via data on $\mathrm{Im}^\pm$ by means of the so called null surface formulation of general relativity. In this peculiar approach it is possible to reproduce up to a conformal factor the metric (satisfying Einstein vacuum field equations) starting from a single function which is a solution of the light cone equation. This function depends both on a bulk point x^a and on the coordinates over S^2 and if we keep x^a fixed it is defined on null infinity and thus it is a boundary data. Using this peculiar formulation of Einstein theory, we will be able to relate boundary and bulk 2-point correlation function in Minkowski background and we will draw as well some conclusions in a generic background.

Outline of the paper. The organization of the paper is the following: in section 2 we review the construction both of the kinematical and of the dynamical data for the classical BMS field theory. Since all the analysis is performed in a momentum frame, in section 3 we switch to the coordinate frame through a peculiar technique proper of infinite dimensional calculus. In section 4 we construct the lagrangian for the BMS field and in particular for the specific case of a massive scalar field; the results from this section allow us to study via path-integral the partition function in section 5 where we also construct the 2-point correlation function. In section 6 we formulate a conjecture to relate the boundary with the bulk data through the formalism of null surface formulation of general relativity. Moreover for sake of completeness there is an appendix on the purely mathematical details of infinite dimensional calculus that we use throughout the paper and there is a second appendix where we review the main idea lying behind the null surface formalism.

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2. BMS fields

In this section we review some of the key results of our previous papers on the subject without entering into unnecessary technical details which are fully described in [7, 8].

Let us consider an asymptotically flat space-time M (see [12] and [13] for a technical definition and analysis); in the so called Bondi reference frame $y_B = (u = t - r, r, \theta, \varphi)$, the future null infinity $\mathrm{Im}^+ \sim \mathbb{R} \times S^2$ can be endowed with a degenerate metric

$$ds^2 = 0 \cdot du^2 + d\Omega^2 \,, \tag{2.1}$$

where $d\Omega^2$ is the solid angle element. Let us briefly remark that both the topological and the metric structure of conformal infinity in an asymptotically flat space-time are universal i.e. there is no trivial physical process which can change (at least at first order) the structure of $\mathrm{Im}^\pm$ providing that the manifold under analysis is flat both at past and future null infinity [14].

Up to a stereographic projection sending the S^2 coordinates (θ, φ) in $(z, \bar{z})$ the diffeomorphism group (preserving the so called "strong conformal structure") of (2.1) is the so called BMS group i.e.

$$u \longrightarrow K(z,\bar{z})\,(u + \alpha(z,\bar{z}))\,, \tag{2.2}$$

$$z \longrightarrow \frac{az+b}{cz+d} \quad ad - bc = 1\,, \tag{2.3}$$

where α is any smooth map from S^2 to the real axis and

$$K = (1+ \mid z \mid)^{-1} \left((az+b)(\bar{a}\bar{z}+\bar{b}) + (cz+d)(\bar{c}\bar{z}+\bar{d})\right)^{-1}. \tag{2.4}$$

From (2.2) and (2.3), it is straightforward to recognise that the composition law for the BMS group has the structure of the semidirect product:

$$BMS_4 = \mathrm{SL}(2,\mathbb{C}) \ltimes N\,,$$

where N (usually called the *supertranslation* subgroup) is the set of maps from S^2 into $\mathbb{R}$ endowed with a suitable topology (see [7] and [15] for a detailed discussion on this point). In particular we will choose from now on the Hilbert topology (i.e. $N = L^2(S^2)$) even though all the results we will achieve apply as well to the nuclear topology i.e. $N = C^\infty(S^2)$ which has been extensively discussed in [16]. Moreover, let us add that there exists a unique four dimensional normal subgroup $T^4 \subset L^2(S^2)$, but this does not allow us to extract a unique Poincaré subgroup from the BMS; on the opposite the number of non equivalent Poincaré subgroups is infinite namely there is one for each element $g \in L^2(S^2)/T^4$ since $g^{-1}(\mathrm{SL}(2,\mathbb{C}) \ltimes T^4)g = \mathrm{SL}(2,\mathbb{C}) \ltimes T^4$.

Since our ultimate goal is to construct a field theory invariant under the action of the BMS group, the first step is to study the spectrum of the fields living on the boundary. This result has been achieved in [7] using the theory of representations for the BMS group developed in [15] and [17] and we will now briefly review it.

A BMS invariant field can be thought as an *induced wave equation* i.e. a map from the orbits in $L^2(S^2)$ of one of the little groups L_χ of BMS_4:

$$\phi^j : \mathcal{O} = \frac{\mathrm{SL}(2,\mathbb{C})}{L_\chi} \hookrightarrow L^2(S^2) \to \mathcal{H}^j\,, \tag{2.5}$$

where $\mathcal{H}$ is a suitable finite-dimensional Hilbert space usually chosen as $\mathbb{C}^j$ where j is the dimension of an irreducible representation of the little group. Each orbit can be described by means of some characteristic labels, namely the Poincaré mass of a particle and the index of a unitary representation of the little group. Since it will be a key concepts in the forthcoming sections, let us briefly remember that the Poincaré mass of a BMS particle is defined through the unique four dimensional abelian normal subgroup T^4 of BMS_4. Let us consider any point $\alpha \in L^2(S^2)$ and its associated character i.e. a map $\chi : L^2(S^2) \to \mathrm{U}(1)$ which assigns to α the phase $\chi(\alpha) = e^{if(\alpha)}$. Through the Riesz-Fischer theorem, we can write $f(\alpha)$ as the product in $L^2(S^2)$ between α and a unique element $p(\theta,\varphi)$. In analogy with a Poincaré invariant theory in a finite dimensional manifold M^d where the character associated to each point $x^\mu \in M^d$ is $\chi(x^\mu) = e^{ip_\mu\cdot x^\mu}$, we can recognise $p(\theta,\varphi)$ as the (super)momentum associated to the point $\alpha \in L^2(S^2)$. Moreover if we expand $p(\theta,\varphi)$ in spherical harmonics we can easily recognise in the first four harmonics the contribution from the normal subgroup T^4. In detail, let us introduce the following projection

$$\begin{aligned} \pi :\ & L^2(S^2) \to T^4 \\ & \pi(p(\theta,\varphi)) = \sum_{l=0}^{1}\sum_{m=-l}^{l} p_{lm} Y_{lm}(\theta,\varphi) \longleftrightarrow p_\mu = (p_{00}, p_{1,-1}, p_{10}, p_{11})\,. \end{aligned} \tag{2.6}$$

which defines the Poincaré mass

$$m^2 = \pi(p(\theta,\varphi)) \cdot \pi(p(\theta,\varphi)) = \eta^{\mu\nu} p_\mu p_\nu\,.$$

Let us now concentrate on the list of BMS little groups ($R_2 = \{I, -I\}$):

Little group	orbit invariant	representation label
SU(2)	$p^2 = m^2$, $\mathrm{sgn}(p_0)$	discrete spin $j(\dim = 2j+1)$
$\Gamma \sim \mathrm{SO}(2)$	$p^2 = m^2$, $\mathrm{sgn}(p_0)$	discrete spin s
Γ	$p^2 = 0$, $\mathrm{sgn}(p_0)$	discrete spin s
Γ	$p^2 = -m^2$	discrete spin s
$\Theta \sim \mathrm{SO}(2)R_2$	$p^2 = m^2$, $\mathrm{sgn}(p_0)$	discrete spin s
discrete groups	$p^2 \gtreqless 0$	finite dim. rep.

This list has to be compared with the corresponding one for a Poincaré invariant theory:

Little group	orbit invariant	representation label
SU(2)	$p^2 = m^2$, $\mathrm{sgn}(p_0)$	spin j
SU(1,1)	$p^2 = -m^2$	discrete spin j'
$E(2)$	$p^2 = 0$, $\mathrm{sgn}(p_0)$	∞-dimensional
$E(2)$	$p^2 = 0$, $\mathrm{sgn}(p_0)$	1-dimensional λ

At first glance we can recognise several differences between the BMS particle spectrum and the Poincaré spectrum; although in both cases there is an SU(2) invariant massive particle, an $E(2)$ massless particle and an unphysical $\mathrm{SU}(1,1)$ particle characterise a Poincaré invariant theory. On the opposite the BMS spectrum shows a plethora of fields related to discrete little groups and massive fields associated to SO(2) and Θ little groups whereas the only possible massless particle comes from the SO(2) representation.

Despite these big differences, we can still recognise that all the Poincaré physical spectrum is fully encoded in the BMS spectrum since, as demonstrated in [8], there is a one to one correspondence between the dynamic (technically speaking between the covariant phase spaces i.e. the space of dynamically possible configurations) of SU(2) BMS and Poincaré fields and more surprisingly between BMS SO(2) and Poincaré $E(2)$ massless fields. Moreover there is also a natural interpretation for the discrete little groups as instantons of the bulk gravitational field. This idea has been introduced in [18] (see also [19] for recent results) but we will not pursue it further in this paper. Nonetheless a correct interpretation at a level of bulk data for the massive SO(2) and Θ fields is still lacking and it is under investigation.

Equation of motion. A further result we can achieve studying the BMS fields through the theory of representation are the equations of motion that characterise the dynamic of each BMS-particle. This result is in some sense the BMS equivalent of Wigner approach to the relativistic wave equations for Poincaré invariant fields; the key step is to introduce the so called *covariant wave equation* i.e. a function

$$\tilde{\phi}^\sigma : L^2(S^2) \to \mathcal{H}^\sigma \,, \tag{2.7}$$

which transforms under the action of the BMS group through a representation of its Lorentz subgroup (more precisely $\mathrm{SL}(2,\mathbb{C})$). The target space $\mathcal{H}^\sigma$ is a finite dimensional Hilbert space usually chosen as $\mathbb{C}^\sigma$ where σ is the dimension of the chosen representation of $\mathrm{SL}(2,\mathbb{C})$. In this picture the equation of motion arise realizing that (2.7), albeit covariant, is an highly redundant wave function which transforms under a non-irreducible representation; thus one has to impose some constraints on (2.7) in order to reduce it to (2.5) which transforms under an irreducible representation of the BMS group. These constraints are twofold; from one side the support of (2.7) has to be reduced from $L^2(S^2)$ to the orbit $\mathcal{O}$ of the little group L_χ whereas, from the other side, one has to act on the target space since the dimension of $\mathcal{H}^\sigma$ in (2.7) is much bigger than the dimension of $\mathcal{H}^j$ in (2.5) i.e. many components of $\tilde{\phi}^\sigma$ are redundant.

It is worth to notice that these operations are all performed in a momentum frame and that for the BMS fields they have been classified and interpreted in [7]. Nonetheless, since in the forthcoming section we will always refer for sake of simplicity to a guiding example, namely the real scalar SU(2) BMS field, let us derive the equations of motion in this setting. The induced and covariant wave functions are respectively:

$$\begin{aligned} \phi :\ & \frac{\mathrm{SL}(2,\mathbb{C})}{\mathrm{SU}(2)} \sim \mathbb{R}^3 \to \mathbb{R}\,, \\ \tilde{\phi} :\ & L^2(S^2) \to \mathbb{R}\,. \end{aligned}$$

As shown in [15], the "orbit constraint" is equivalent to the vanishing of the pure super-translational part in the supermomentum i.e.

$$p(\theta,\varphi)-\pi(p(\theta,\varphi))=0\,. \tag{2.8}$$

This equation reduces the dynamical space from $L^2(S^2)$ to $\mathbb{R}^4$. Furthermore, also the mass equation has to be imposed i.e.

$$[\pi(p(\theta,\varphi))\cdot\pi(p(\theta,\varphi))-m^2]\phi(p)=0\,, \tag{2.9}$$

which selects the orbit $\frac{SL(2,\mathbb{C})}{SU(2)}$ embedded in $\mathbb{R}^4$. The set (2.8) and (2.9) determine the full dynamic of a BMS scalar field and we will refer to them as the *BMS-Klein-Gordon* equations. No further condition has to be imposed on the target space since the dimension is the same in the induced and covariant approach.

In order to convince the reader that the result is meaningful, let us briefly repeat the construction for the Poincaré scalar field. In this scenario the induced wave function has the same expression as in the BMS case:

$$\psi:\frac{SL(2,\mathbb{C})}{SU(2)}\sim\mathbb{R}^3\to\mathbb{R}\,,$$

whereas the covariant function is

$$\tilde{\psi}:T^4\to\mathbb{R}\,.$$

The only constraint that has to be imposed is the mass equation:

$$\left[p^\mu p_\mu-m^2\right]\tilde{\psi}(p)=0\,,$$

which is exactly the Klein-Gordon equation in the momentum frame.

A key feature of Wigner's approach in order to deduce the equations of motion of a Poincaré (or a BMS) particle is the total absence either of a lagrangian or of an hamiltonian function to minimise in order to describe the dynamic. This aspect is highly interesting since it shows that the dynamic of a free particle is purely a group-theoretical result but an hamiltonian (or equivalently a lagrangian) approach is nonetheless essential in order to construct the quantum theory from a path-integral point of view. In the BMS scenario this problem has been partially solved in [8] where the free hamiltonians for each BMS field has been constructed. Considering always the leading example of the scalar field let us review the key aspects and results of the construction. In particular, let us notice that the key ingredient in an hamiltonian framework is the correct identification of a phase space; in the usual approach to quantum field theory this is achieved slicing the 4-dimensional manifold as $\Sigma_3\times\mathbb{R}$ and recognising $\mathbb{R}$ as the evolution direction and Σ_3 as the Cauchy surface where initial data are defined. Thus the canonical phase space associated to the dynamical theory of a covariant field ϕ is usually identified as the set $\Gamma=\{(\varphi,\pi)\}$ where $\varphi=\phi|_{\Sigma_3}$ and $\pi=\frac{\delta\mathcal{L}}{\delta\phi}\,|_{\Sigma_3}$. This approach is not pursuable in the BMS case since we lack the identification both of a lagrangian and of a suitable initial Cauchy surface in $L^2(S^2)$. Nonetheless an alternative road to define an hamiltonian is to consider the so-called *covariant phase space* which is the set of fields satisfying the equations of motion i.e.

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it is the space of all dynamically possible configurations. This space is smaller[2] than the canonical phase space and we refer to the literature ([20] and references therein) for the reader interested in its main characteristics.

For a BMS scalar field the covariant phase space is:

$$\Gamma|_{\rm cov} = \left\{ \phi : \frac{\mathrm{SL}(2,\mathbb{C})}{\mathrm{SU}(2)} \to \mathbb{R} \ \ [\pi(p)\cdot\pi(p) - m^2]\phi(p) = 0 \right\}. \tag{2.10}$$

Since the constraints defining the equations of motion for a free field are first class, we can identify $\Gamma|_{\rm cov}$ as a linear space which can be endowed with a suitable regularity condition requiring either that the fields are smooth i.e. of class C^∞ or that they are square-integrable respect to a suitable measure. In both cases we end up with a covariant phase space which is a Banach or an Hilbert space and, thus, we can apply a theorem from Marsden and Chernoff (see [21] and also [22] for an earlier application in a BMS scenario) which grants us that, if we can endow (2.10) with a symplectic structure Ω, any Lie algebra generator ζ of $\frac{\mathrm{SL}(2,\mathbb{C})}{\mathrm{SU}(2)}$ is globally hamiltonian with energy function

$$H_\zeta(\phi) = \frac{1}{2}\Omega(\mathcal{L}_\zeta\phi, \phi)\,, \tag{2.11}$$

where $\mathcal{L}_\zeta$ is the Lie derivative along the ζ generator. Let us further remark that a choice for the symplectic structure on $\Gamma|_{\rm cov}$ can be performed reminding that $\mathcal{O} \sim \frac{\mathrm{SL}(2,\mathbb{C})}{\mathrm{SU}(2)}$ is equivalent to $\mathbb{R}^3$ endowed with the hyperbolic metric. Thus we can write the canonical

$$\Omega(\phi,\phi') = \int_{\mathcal{O}} d\mu(\mathcal{O})\left(\phi\nabla\phi' - \phi'\nabla\phi\right),$$

where ∇ is the covariant derivative.

3. From momentum frame to the coordinate frame: an infinite dimensional calculus

As we have explained in the previous section, the pure group theoretical approach has allowed us to write the equations of motion for a BMS field in the momentum frame. Since the aim is to derive these equations from a variational principle, the first step is to write them in a coordinate frame. Nonetheless, at first glance, this task is rather challenging since the space of variables under consideration is not finite-dimensional and thus we cannot perform the usual Fourier transform straightforward. In order to circumvent this problem, we have to realize that the configuration space of fields can be endowed with some regularity constraint namely we can require it to be a Banach or an Hilbert space. In particular from now on we will choose it as:

$$\mathcal{C} = \left\{ \phi : N = L^2(S^2) \to \mathbb{C}^\lambda \mid \phi \in L^2(N)\otimes\mathbb{C}^\lambda \right\}.$$

[2]There is only one exception i.e. the Poincaré scalar field where the covariant and the canonical phase space are in one to one correspondence.

This requirement allow us to activate several interesting techniques which have been developed in the study of white noise analysis as an infinite dimensional calculus. In particular the main results we are interested in are, from the pure mathematical point of view, an extension to an infinite dimensional setting of key Hilbert space techniques such as Fourier transform and distribution theory. Since these results are fundamental to this paper, we will review them to a certain extent in the appendix following [23] and [24]. For the less mathematically oriented reader, we wish to emphasise that, in the BMS scenario, the role of the finite dimensional manifold $\mathbb{R}^n$ is played by $L^2(S^2)$ which is a riemannian infinite-dimensional manifold endowed with a positive definite metric defined through the usual internal product. Thus, for our purposes, the space of square integrable function over $L^2(S^2)$ and the operators acting on it behave exactly as in $L^2(\mathbb{R}^n)$ except for some peculiar aspects which we will point out as soon as we need them.

Thus, let us start considering the BMS equations (2.8) and (2.9) for the scalar field in a momentum frame.

Theorem 1. *The support condition $[p(\theta,\varphi)-\pi(p)]\phi(p)=0$ is equivalent to*

$$Q_{p-\pi(p)}\phi(p)=0\,, \tag{3.1}$$

where ϕ is a non vanishing function in $\mathcal{C}$ and where $Q_{p-\pi(p)}$ is the multiplication operator in $L^2(N)$ along the element[3] $p-\pi(p)$ in $N=L^2(S^2)$. At the same time, if we introduce the basis in N, $\{e_i\}=\{Y_{00},Y_{1-1},Y_{10},Y_{11},\ldots\}$, equation (3.1) is equivalent to

$$Q_{e_i}\phi(p)=0 \qquad i=4,\ldots,\infty\,.$$

In order to demonstrate the above theorem let us remark that, if we expand in spherical harmonics the supermomentum $p(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}p_{lm}Y_{lm}(\theta,\varphi)$, the support condition implies $p_{lm}=0$ for $l>1$. The action of the multiplication operator is (see appendix A for the definition)

$$Q_{p-\pi(p)}\phi(p)=[\langle p,p\rangle_N-\langle p,\pi(p)\rangle_N]\phi(p)=0\,,$$

where $\langle,\rangle_N$ is the canonical internal product on N. Using the orthogonality relations between the spherical harmonics and the definition of the projection π, the above formula implies

$$\sum_{l>1}\sum_{m=-l}^{l}\mid p_{lm}\mid^2=0\,.$$

Since this is a sum of positive quantities, this statement is equivalent to $p_{lm}=0$ for any $l>1$ which is exactly the support condition for a BMS SU(2) field. At the same time the request $p_{lm}\phi(p)=0$ for a non vanishing function $\phi(p)$ can be written as:

$$\langle Y_{lm},p\rangle\phi(p)=Q_{Y_{lm}}\phi(p)=0\,,\qquad l>1\,.$$

This implies the statement $Q_{e_i}\phi(p)=0$ with $i>4$.

[3]From now on the dependence on the S^2 angular variable is implicit unless stated.

Equivalently the mass equation for a BMS scalar field can be written through the multiplication operators. Let us put (2.9) in the form

$$\left[\eta^{\mu\nu}\pi(p)_\mu\pi(p)_\nu - m^2\right]\phi(p) = 0\,, \tag{3.2}$$

where $\eta^{\mu\nu}$ is the flat lorentzian metric and $\pi(p)_\mu = \langle\pi(p), e_\mu\rangle$ i.e. it is the projection of the T^4 part of the supermomentum along the direction in $L^2(S^2)$ defined by the first four harmonics. Let us stress that the explicit presence of the metric η is needed since the above equation characterise the mass hyperboloid embedded in the riemannian space $L^2(S^2)$; thus, there is no way to write the above relation only using the natural riemannian internal product on the configuration space and a second lorentzian scalar product has to be introduced.

Theorem 2. *The mass equation (3.2) is equivalent to:*

$$[\eta^{\mu\nu}Q_{e_\mu}Q_{e_\nu} - m^2]\phi(p) = 0\,, \tag{3.3}$$

where $\{e_\mu\}$ is the set of first four harmonics.

In order to demonstrate this theorem, we only need to notice that the following chain of identities holds:

$$\pi(p)_\mu = \langle\pi(p), e_\mu\rangle = \langle p, e_\mu\rangle\,.$$

Thus the mass equation can be written as:

$$\left[\eta^{\mu\nu}\langle p, e_\mu\rangle\langle p, e_\nu\rangle - m^2\right]\phi(p) = [\eta^{\mu\nu}Q_{e_\mu}Q_{e_\nu} - m^2]\phi(p) = 0\,,$$

where in the second equality we have applied the definition of multiplication operator acting on $L^2(N)$ with $N = L^2(S^2)$.

We have now written the equation of motion of the Klein-Gordon BMS field through the action of suitable multiplication operators acting on the covariant wave function in the momentum frame. The last step consists on using the properties of the Fourier transform in infinite dimensions in order to switch to the coordinate frame. Leaving the mathematical details and definitions to the appendix, let us only stress that the Fourier transform, although defined in a non intuitive way, shares the same properties with its finite-dimensional counterpart when acting on operators. Thus, calling with $\mathcal{F}$ the Fourier transform, the following identities hold:

$$\mathcal{F}Q_\eta = iD_\eta\,,$$
$$\mathcal{F}D_\eta = iQ_\eta\,,$$

where D_η is the Gateaux (i.e. directional) derivative along η. Thus, using these relations, we can write the BMS Klein-Gordon equation in the coordinate frame:

$$\left[\eta^{\mu\nu}D_{e_\mu}D_{e_\nu} + m^2\right]\hat{\phi}(x) = 0\,, \tag{3.4}$$
$$D_{p-\pi(p)}\hat{\phi}(x) = 0\,, \tag{3.5}$$

where $x \in L^2(S^2)$, $\hat{\phi}$ is the Fourier-transformed covariant wave function and where the second equation implies the vanishing of the gradient of ϕ along the pure supertranslational directions. This relation can be decomposed in its components as:

$$D_{e_i}\hat{\phi}(x) = 0\,, \qquad i > 4\,. \tag{3.6}$$

Thus, we have achieved our goal to write the equation of motion for a scalar field in the coordinate frame but a natural question that arises is how general is this construction and how far we can apply it to all other BMS fields. The answer is that the above construction is completely general and let us make some comments and remarks on this issue.

First of all let us remember that, given a covariant wave function $\phi^\lambda : \mathcal{O} \sim \frac{\mathrm{SL}(2,\mathbb{C})}{L_\chi} \to \mathbb{C}^\lambda$, the equations of motion in momentum frame are:

$$\begin{cases} p - G\bar{p} = 0\,, \\ \left[\pi(p)\cdot\pi(p) - m^2\right]\phi^\lambda(p) = 0\,, \\ \rho^\lambda(p)\phi^\lambda(p) = \phi^\lambda(p)\,, \end{cases} \tag{3.7}$$

where $G\bar{p}$ is the action of the full $\mathrm{SL}(2,\mathbb{C})$ group on the fixed point $\bar{p}$ in $L^2(S^2)$ associated to the little group L_χ [15]. The term ρ^λ is a matrix reducing the redundant components of the wave function in $\mathbb{C}^\lambda$ (see [7] for an explicit characterisation and construction in the BMS scenario). If we want to write these equations in the coordinate frame, we can immediately draw some conclusions: the mass equation is identical for every BMS field (both with vanishing or non vanishing mass), thus (3.4) can be thought as a general equation holding for any little group. This is not surprising if we think to the Poincaré case where the mass equation is simply the Klein-Gordon equation which holds for every covariant field.

The orbit constraint instead is dependent on the chosen little group L_χ. Thus we can immediately conclude that (3.5) holds for any SU(2) BMS particle. Moreover, if we change the little group, one can immediately notice without big technical problems that equation (3.5) becomes in a coordinate frame

$$D_{p-G\bar{p}}\hat{\phi}^\lambda(x) = 0\,,$$

which grants us that the gradient of the wave function vanishes outside the orbit of the classical motion. As a side remark, we can also notice that this equation can be written in the form

$$D_{e_i}\hat{\phi}(x) = 0\,, \qquad i > \dim\frac{\mathrm{SL}(2,\mathbb{C})}{L_\chi} + 1\,, \tag{3.8}$$

providing that we perform a suitable non canonical choice of a base in $L^2(S^2)$. The last term in (3.7) has not been discussed in the Klein-Gordon scenario since it represents the dynamic of the field on the orbit which, in the scalar case, is fully encoded in the mass equation. Moreover, this term is non universal and it encodes the differences between the BMS fields; nonetheless we can perform the same tricks as for the orbit and the mass term in order to write it in the coordinate frame. To better clarify this point let us consider the spin $\frac{1}{2}$ BMS field; if we take into account the first four harmonics and if we follow in a straightforward manner the details of the calculation in [25], the dynamical equation on the orbit can be written as:

$$[\gamma^\mu p_\mu - m]\phi(p) = 0\,,$$

where γ^μ are the Dirac matrices and where $p_\mu = \langle e_\mu, p\rangle$ (e_μ being one of the first four harmonics). Thus, in the language of operators, this equation is

$$[\gamma^\mu Q_{e_\mu} - m]\phi(p) = 0\,.$$

Performing a Fourier transform, this equation becomes

$$[i\gamma^\mu D_{e_\mu} - m]\hat{\phi}(x) = 0\,.$$

Thus, the form of the equation of motion for a BMS particle can be written in the coordinate frame using infinite dimensional techniques and the expression is rather similar, at least in the SU(2) case, to the Poincaré expression. From a physical point of view, this is not surprising since the main difference between the Poincaré and the BMS approach comes from the pure supertranslational part which is at classical level sterile since (3.5) grants us that along these directions the covariant field is constant. This confirms the result, already achieved in [8], that the dynamic of Poincaré fields is fully encoded in the BMS case and there is a complete equivalence if we consider the SU(2) fields. Nonetheless let us anticipate that the pure supertranslational term, although only a constraint at a classical level, will play a fundamental role at a quantum level when performing path-integration on the full space of configuration and not only on the covariant one. A last remark that applies to the SU(2) BMS fields is related to the nature of the Poincaré subgroups of the full BMS group. As we have previously discussed the latter has a unique four dimensional subgroup but nonetheless we cannot single out a unique Poincaré subgroup. In particular, as we have shown in the previous section, the number of different Poincaré subgroups is infinite i.e. equal to the number of elements in $L^2(S^2)/T^4$. At a level of classical dynamics, this feature completely disappears since the orbit of SU(2) BMS-particles is characterised by a vanishing pure supertranslation selecting in some sense a unique Poincaré subgroup associated to the identity element in $L^2(S^2)/T^4$. We will comment further on this issue in section 6 when, addressing the problem of quantisation, the presence of different Poincaré subgroups will play a leading role also in the SU(2) case.

4. The BMS lagrangian

Bearing in mind the results of the previous section, we are now entitled to investigate from which lagrangian these equations of motion arise. Let us consider the BMS Klein-Gordon field and let us start from (3.4) and (3.5). As underlined before, (3.5) is acting as a constraint on the wave function and it does not provide any dynamic; thus, the function we are looking for has to be divided in two parts: the first which takes into account the mass term and the dynamic of the scalar field and the second which takes into account the constraints through a Lagrange multiplier. In an infinite dimensional setting the theory of

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Lagrange multiplier has been studied in [26] and, using these results, we can introduce the following lagrangian:

$$L[\phi] = \phi(x)\left[\eta^{\mu\nu} D_{e_\mu} D_{e_\nu}\phi(x) - m^2\phi(x)\right] + \sum_{i=4}^{\infty}\gamma_i(x) D_{e_i}\phi(x)\,, \tag{4.1}$$

which gives us the original equation of motion and where $\gamma_i(x)$ are the Lagrange multipliers. As a side remark and for sake of completeness, we wish to point out that the support of each multiplier is for simplicity the whole Hilbert space $L^2(S^2)$; this does not grant that the minimum problem in (4.1) has a unique solution whereas this would have been avoided if $\gamma_i(x)$ were defined on the orbit of the little group. Since this issue does not play a key role in our analysis, we will not refer to this question anymore leaving to [26] for more details.

Following the definition of the lagrangian, we can introduce as well the gaussian measure $d\mu$ in the Hilbert space $L^2(S^2)$ (see [27] for a definition and properties) in order to write an action

$$S[\phi] = S_{\rm dyn} + S_{\rm constr}\,, \tag{4.2}$$

$$S[\phi] = \int_{L^2(S^2)} d\mu(x)\,\phi(x)\left[\eta^{\mu\nu} D_{e_\mu} D_{e_\nu}\phi(x) - m^2\phi(x)\right] + \sum_{i=4}^{\infty}\gamma_i(x) D_{e_i}\phi(x)\,. \tag{4.3}$$

Let us emphasise a key difference between (4.3) and its finite dimensional Poincaré-invariant counterpart namely that, whereas through the Stokes theorem the Klein-Gordon action can be written as

$$S(\psi) = \int_{M^4} d^4x\sqrt{\mid g\mid}\left[\partial^\mu\psi\partial_\mu\psi - m^2\psi^2\right],$$

this cannot be done in the BMS case. As a matter of fact, the dynamical part in (4.3) can be written as an internal product over the space of fields:

$$S_{\rm dyn}[\phi] = \langle\phi, \eta^{\mu\nu}(D_{e_\mu} D_{e_\nu})\phi\rangle - m^2\langle\phi,\phi\rangle\,.$$

Using the definition of adjoint operator and the relation $D^*_\eta = Q_\eta - D_\eta$ (see appendix A for a formal definition), the above expression becomes:

$$\begin{aligned} S_{\rm dyn}[\phi] &= \eta^{\mu\nu}\langle D^*_{e_\mu}\phi, D_{e_\nu}\phi\rangle - m^2\langle\phi,\phi\rangle \\ &= \eta^{\mu\nu}(-\langle D_{e_\mu}\phi, D_{e_\nu}\phi\rangle + \langle Q_{e_\mu}\phi, D_{e_\nu}\phi\rangle) - m^2\langle\phi,\phi\rangle\,, \end{aligned}$$

which, up to a sign, implies:

$$\begin{aligned} S[\phi] = \int_{L^2(S^2)} d\mu\Bigg[&\eta^{\mu\nu}\left(D_{e_\mu}\phi(x) D_{e_\nu}\phi(x) - \langle e_\mu, x\rangle\phi(x) D_{e_\nu}\phi(x) +\right) + \\ &+ m^2\phi^2(x) + \sum_{i=4}^{\infty}\gamma_i(x) D_{e_i}\phi(x)\Bigg]\,. \end{aligned}$$

As for the equations of motion, a natural question arising from the study of the Klein-Gordon action is how general is this construction and how far it can be applied to any other free field. In order to answer to this question we need first of all to separate the contributions of the Lagrange multipliers from the pure dynamical term. The action

$$S_{\text{constr}}(\phi) = \int_{L^2(S^2)} d\mu(x) \left[\sum_{i=4}^{\infty} \gamma_i(x) D_{e_i} \phi(x) \right]$$

is a general term which is dependent only on the little group and, thus, the above expression holds for any BMS SU(2) field. Moreover similar expressions can be written for any little group starting from the equation of motion in the form of (3.8) whereas the tricky point is the dynamical part of the action. The latter changes for each different field we consider and a general expression cannot be written. Nonetheless the recipe we used in the scalar field scenario is completely general and it can be adapted to any other field providing that an expression for the equation of motion in local coordinates over the orbit is found.

Massless particles. Let us now consider more in detail the case of massless particles that, in a BMS setting, are associated to $\Gamma \sim SO(2)$ induced wave functions i.e. if we take a covariant wave function $\psi^\lambda : L^2(S^2) \to \mathbb{C}^\lambda$ the equations of motion are:

$$\begin{cases} [p - G\bar{p}(\theta)]\psi^\lambda(p) = 0\,, \\ \pi(p) \cdot \pi(p)\psi^\lambda(p) = 0\,, \\ \rho^\lambda(p)\psi^\lambda(p) = \psi^\lambda(p)\,, \end{cases} \tag{4.4}$$

where the fixed point $\bar{p}$ is a function only of the θ-variable and $\rho^\lambda(\bar{p})$ is a diagonal matrix with all 0 entries except the first equal to 1.

Since we are interested in an holographic description of bulk data, we are entitled to restrict our analysis to a specific orbit of the Γ little group with vanishing supertranslation. As we have shown in [7], the possible orbits in this peculiar scenario are characterised by the choice of the "energy" for the massless field i.e. the p_0 component of $\pi(p)$ and of a pure supertranslation function[4] $\alpha(\theta)$. Nonetheless the value $\alpha = 0$ has a peculiar meaning since, in this case, it is possible to show (see [8]) that the covariant phase space of a BMS massless SO(2) particle is in one to one correspondence with the covariant phase space of a Poincaré massless $E(2)$ particle; in plain words, this implies that the classical dynamic of a massless bulk field in a Minkowski background is fully equivalent to the boundary SO(2) BMS counterpart. Thus, in the peculiar setting, $\alpha = 0$ (4.4) reduces to

$$\begin{cases} [p - \pi\bar{p}(\theta)]\psi^\lambda(p) = 0\,, \\ \pi(p) \cdot \pi(p)\psi^\lambda(p) = 0\,, \\ \rho^\lambda(p)\psi^\lambda(p) = \psi^\lambda(p)\,. \end{cases} \tag{4.5}$$

Moreover, if we consider the massless real (or complex) scalar field i.e. with vanishing helicity, we deal with a one component field ($\psi^\lambda : L^2(S^2) \to \mathbb{R}$) and $\rho^\lambda(p) = 1$. Thus, the last equation is identically satisfied and (4.5) is equal to (2.8) and (2.9) with the due exception that $m = 0$. This implies that the analysis performed for the massive case can

[4] In the literature, a pure supertranslation refers to an element of the group $L^2(S^2)/T^4$ i.e. a real function on S^2 depending on spherical harmonics with $l > 1$.

be extended as well to the SO(2)-massless field and from now on all the results achieved will apply as well to the massless scenario (i.e. considering simply the limit $m \to 0$) unless stated otherwise.

5. Path-integral and BMS correlators

In the previous section we have explicitly constructed the lagrangian of a scalar BMS particle and we have given a recipe to perform the same calculation for any other field. This result allow us to address a key question from the holographic point of view: what are the main features of the boundary theory at a quantum level, what is the explicit expression of the correlation functions and what is eventually their relation (if it exists) with the bulk counterpart. Seeking an answer to these questions, we will discuss a path-integral approach to the BMS free fields always through the leading example of the scalar particle. Thus, starting from (4.3), we need to evaluate:

$$Z[\phi,\gamma] = \int_{\mathcal{C}} e^{iS[\phi,\gamma]} d[\phi,\gamma]\,,$$

where $\mathcal{C}$ is the set[5] of possible kinematical configurations endowed with some regularity condition. Remembering that the action is split in $S = S_{\rm dyn} + S_{\rm constr}$ and that the fields $\gamma_i(x)$ act only as Lagrange multiplier, we can use the canonical relation due to Faddeev [28] in order to eliminate $\gamma_i(x)$:

$$Z[\phi] = \int_{\mathcal{C}} d[\phi]\delta\left(\sum_{i=4}^{\infty} D_{e_i}\phi(x)\right) \exp i\left[\phi(x)\left[\eta^{\mu\nu}D_{e_\mu}D_{e_\nu}\phi(x) - m^2\phi(x)\right]\right].$$

The infinite dimensional delta is reminiscent of those appearing in gauge theory when performing the Faddeev-Popov trick in order to take into account the gauge degrees of freedom. Thus, we substitute the delta function with the following combination:

$$Z[\phi] = C\int_{\mathcal{C}} d[\phi]e^{iS_{\rm dyn}} \sum_{i>4}\int d[\omega] e^{\frac{i}{2\zeta_i}\langle\omega_i(x),\omega_i(x)\rangle_N} \delta(D_{e_i}\phi(x) - \omega_i(x))\,,$$

where C is an irrelevant constant, $\langle,\rangle_N$ is the internal product in $L^2(S^2)$ and ζ_i is a non divergent real number. Thus, performing the integration over the ω-functions, we end up with

$$Z[\phi] = \int_{\mathcal{C}} d[\phi]e^{iS_{\rm eff}}\,,$$

where

$$S_{\rm eff} = S_{\rm dyn} + \int_{L^2(S^2)} d\mu(x) \sum_{i>4}\left[\frac{1}{2\zeta_i}D_{e_i}\phi(x)D_{e_i}\phi(x)\right]. \tag{5.1}$$

In order to evaluate the path-integral for a free field we have to seek if it is possible to reduce the action in a form $S(\phi) = \langle\phi, B\phi\rangle_N$. If we look at (5.1) we see that only the

[5] We wish to emphasise to the reader that the set $\mathcal{C}$ is not necessarily equivalent to the set of scalar fields defined in the previous section in order to construct the lagrangian

second term has to be changed. In particular each element in the sum can be written as

$$\begin{aligned} S' &= \frac{1}{2\zeta_i}\langle D_{e_i}\phi(x), D_{e_i}\phi(x)\rangle_N \\ &= \frac{1}{2\zeta_i}\langle \phi(x), D^*_{e_i}D_{e_i}\phi(x)\rangle_N \\ &= \frac{1}{2\zeta_i}\langle \phi(x), (Q_{e_i}-D_{e_i})D_{e_i}\phi(x)\rangle_N \,, \end{aligned}$$

where we have used the definition of adjoint operator and the relation $D^*_\eta + D_\eta = Q_\eta$. Thus the operator B is

$$B = \eta^{\mu\nu}D_{e_\mu}D_{e_\nu} + m^2 + \sum_{i>4}\frac{1}{2\zeta_i}(Q_{e_i}-D_{e_i})D_{e_i}\,. \tag{5.2}$$

In analogy with the results for the Poincaré free fields, we can formally evaluate $Z[\phi]$ up to a constant as a functional determinant:

$$Z[\phi] = \mathrm{const}\cdot[\det B]^{-\frac{1}{2}}\,.$$

Let us stress that the above result already shows how the supertranslations act in a rather peculiar way at a quantum level. Although at a classical level they are sterile, the functional determinant for the path-integral is not given only by the Poincaré part but the enforcing of the constraint for a vanishing supertranslation implies the presence of an additional term which contains surprisingly the multiplication operator.

We can now address the main question in our investigation: the correlation functions. The first object we need to calculate is the two point function i.e.

$$D(x_1-x_2) = \langle T[\phi(x_1)\phi(x_2)]\rangle = \lim_{T\to\infty}\frac{\int d[\phi]\phi(x_1)\phi(x_2)\exp\left[i\int_{-T}^{T}d\mu L\right]}{\int d[\phi]\exp\left[i\int_{-T}^{T}d\mu L\right]}\,.$$

In the BMS scenario the time parameter is not natural to choose as in a finite-dimensional theory but nonetheless, remembering the construction of the hamiltonian functions, we can use the evolution parameter on the orbit as a natural candidate.

Explicitly, the propagator (or the 2-point correlator) can be calculated as the inverse of the B operator [28] i.e. in a coordinate frame

$$BD(x_1-x_2) = i\delta(x_1-x_2)\,,$$

where x_1, x_2 are two points in $L^2(S^2)$ and where the right hand side is, up to the pure imaginary constant, the infinite-dimensional delta [24]. Performing the Fourier transform, we end up with

$$\left[\eta^{\mu\nu}Q_{e_\mu}Q_{e_\nu} - m^2 + \sum_{i>4}\frac{1}{2\zeta_i}(D_{e_i}-Q_{e_i})Q_{e_i}\right]\hat{D}(k) = i\,. \tag{5.3}$$

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Choosing the value of ζ_i in such a way that $\sum_{i>4} \frac{1}{2\zeta_i} = 0$, the above equation becomes

$$\left[\eta^{\mu\nu}k_\mu k_\nu - m^2 + \sum_{i>4} \frac{1}{2\zeta_i}[-k_i^2 + k_i D_{e_i}]\right] \hat{D}(k) = i\,. \tag{5.4}$$

Eventually we have a formal expression for the 2-point correlation function in the momentum frame and surprisingly we end up with a first-order differential equation. Let us make some comments:

- The (differential) equation for the propagator can be divided in two parts: the first is identical to the Poincaré contribution to the usual 2-point function although the coefficients k_μ are not a priori related to momenta in the bulk and the second comes from the constraints i.e. it is a pure supertranslational effect. Thus we can claim that the result is rather general since $\sum_{i>4} \frac{1}{2\zeta_i}[-k_i^2 + k_i D_{e_i}]$ will appear in all the SU(2) BMS fields and the $\eta^{\mu\nu}k_\mu k_\nu - m^2$ will be substituted with the Poincaré counterpart for any other SU(2) field.

- since (5.4) is a differential equation, we need to assign a suitable initial condition. At a geometric level it is rather interesting to notice that the initial surface is isomorphic to T^4 whereas the pure supertranslational directions (i.e. $L^2(S^2)/T^4$) act as the evolution directions. Thus the initial condition has to be assigned on a four-dimensional surface and the most natural one is to impose that the propagator has a Poincaré-type form for a fixed value of the supertranslation i.e.

$$\hat{D}(k) = \frac{i}{\eta^{\mu\nu}k_\mu k_\nu - m^2}\,, \quad k_i = \bar{k}_i \qquad D_{e_i} D\,(k)\,|_{k_i=\bar{k}_i} = \bar{k}_i D(k)|_{k_i=\bar{k}_i}\,.$$

At the end of our calculation we have the following result: the two point function of a BMS field obeys in the momentum frame a differential equation and the pure supertranslational directions, sterile at a classical level, are the main responsible for this behavior. Thus, solving (5.4), gives us, roughly speaking, the evolution of the propagator from one T^4 to another or, remembering that there are as many equivalent Poincaré fields as pure supertranslations, from one specific Poincaré group to a second one. This arises several interesting question; at an holographic level clearly it is imperative to understand how to reconstruct the bulk information form the boundary theory. Let us stress that the role of $\mathrm{Im}^{\pm}$ is universal thus the theory living on the boundary is the same for any asymptotically flat space-time and the BMS field theory cannot distinguish a priori from one specific metric to another. We will propose a possible solution to these questions in the forthcoming section. Moreover several interesting analogies arise from the BMS field theory (at least for SU(2) fields) compared to Ashtekar's program of asymptotic quantization [30]; in particular the main variables Ashtekar proposes to be quantised are the so called Q_{ab}-variables which are the integrals along the degenerate direction on Im of the news tensor:

$$Q_{ab}(\theta,\varphi) = \int_{\mathrm{Im}} d^3x N_{ab}(u,\theta,\varphi)\,.$$

The interesting aspect of this approach is that each Q_{ab} carries the information of a different infrared sector of the gravitational field and, naively speaking, these are the functions bringing from one Poincaré subgroup to another i.e. they are pure supertranslations. This is exactly the behavior which appears in our propagator i.e. it is a differential equation bringing us from one T^4 (i.e. a Poincaré subgroup) to another and, thus, we could think to each T^4 in (5.4) as labeled by a different value of Q_{ab} opening as well the chance to study Ashtekar variable in terms of operators on an infinite dimensional Hilbert space namely $L^2(L^2(S^2))$. Further relations between IR sectors of a pure radiative gravitational system and the BMS group are currently under investigation [31]

6. From boundary to bulk: a proposal

In order to understand if there is a possible holographic relation between a BMS field theory and the bulk counterpart, the main task is to relate the correlation functions constructed in the previous section with those living in the bulk.

Let us stress that the contents of this section are to a certain degree speculative and let us start with some remarks: as mentioned before, the BMS group is the diffeomorphism group for Im^+ which is a universal structure i.e. any asymptotically flat space-time has the same boundary-structure. This property extends as well at a level of field theory and correlators, such as (5.4), do not distinguish between two different bulk manifolds. Nonetheless the BMS-invariance of the boundary theory is a reasonable request since it is difficult to conceive a quantum field theory that is not invariant under a diffeomorphism transformation. Thus the path followed until now is essentially a natural one, but our construction faces some difficulties: if we look at (5.4), we can recognise that, even though we could exactly solve the infinite-dimensional differential equation, we would face the following questions:

- The coefficients appearing in the propagator (i.e. $k_i = \langle e_i, k >)$ are orthoprojections along spherical harmonics and they have no a priori direct natural relation with coordinates of a bulk reference frame. Nonetheless, if we remember that in a subclass of asymptotically flat space-times such as Minkowski, the BMS group can univocally be reduced to a Poincaré group, (5.4) with $k_i = 0$ (vanishing supertranslation) is formally identical to the scalar field propagator in flat background thus suggesting a deeper relation between (5.4) and the bulk counterpart,

- BMS fields are defined as maps from $L^2(S^2)$ in a suitable target vector space V. This definition has been a natural choice since we have followed Wigner approach to construct the particle spectrum, the equations of motion and the (free field) hamiltonians from a pure group theoretical point of view. In the Poincaré scenario the corresponding choice is [29]
$$\phi^\lambda : T^4 \sim \mathbb{R}^4 \to V\ .$$
This definition is globally correct only in a Minkowski space where the underlying variety is $\mathbb{R}^4$ endowed with the flat metric η^{ij}. In a general framework, the manifold

M^4 (endowed with a generic lorentzian metric g^{ij}), where the fields live, is modeled on $\mathbb{R}^4$ and thus the above definition fits only locally. The same problem applies for the BMS case where $L^2(S^2)$ in the natural counterpart of T^4 but there is no counterpart for M^4 i.e. we lack a natural choice for a global manifold modeled on $L^2(S^2)$ acting as a configuration space.

- in the construction of (5.4) there is no notion of the bulk metric g^{ij}. From one side this is a natural consequence of the universality of $\mathrm{Im}^{\pm}$ and of the BMS group, whereas, from the other side, from the holographic point view it is imperative to retrieve information on a non trivial bulk metric. Let us emphasise that this is a tricky point since we should also take into account that the presence of $\eta^{\mu\nu}$ in (5.4) is a natural choice for a theory living on the boundary of an asymptotically flat space-time where the metric is indeed flat.

We will now propose a possible solution for the above mentioned problems through techniques and results developed in the null surface formulation of general relativity. Since we review its main characteristics in appendix B, we strongly suggest the reader who is not familiar with this formulation of Einstein theory to read the appendix before. Let us simply remember that, within this approach, the (conformal) information of a metric in a 4-dimensional manifold M^4 can be reconstructed starting from a characteristic function [32]

$$u = Z(x^a, z, \bar{z})\,, \tag{6.1}$$

where x^a is a point lying in M^4, $(u, r, z, \bar{z})$ is a Bondi coordinate system in the neighbourhood of Im and (6.1) is the solution in the variable u of the null cone equation $L(x_a, x'_a) = 0$ where $x'_a \in \mathrm{Im}$. As explained in detail in appendix B, (6.1) encodes all the information of the metric in the bulk manifold M^4 and it can be used together with a scalar function $\Omega : M^4 \times S^2 \to \mathbb{R}$ as key ingredients in an alternative formulation of Einstein's equations. Moreover $Z(x^a, z, \bar{z})$ suits for different interpretations: for an arbitrary but fixed value of the pair $(z, \bar{z})$, it represents a null surface with respect to the metric in M^4 whereas, if x^a is kept fixed, it represents real a 2-surface lying on $\mathrm{Im}^{\pm}$. Thus, within this latter framework, (6.1), also called *cut function* $\mathcal{C}_x$, analytically describes the intersection of the light cone with apex x^a with Im^+ and it is a boundary data. Moreover, since these functions are constructed inverting the light cone equation, there exists always a suitable open set near Im^+ where $Z(x^a, z, \bar{z})$ is a smooth function and, thus, it lies in $C^\infty(S^2)$ (or $L^2(S^2)$ if we introduce a further integrability condition) although in general they are globally neither differentiable neither single-valued. These remarks suggest us to adopt a new point of view in our analysis of a boundary BMS-invariant field theory choosing the set of cut functions (parametrically dependent on a bulk point x_a) as the configuration space;[6] this also implies a solution for the second of the problems outlined at the beginning of the section.

Behaviour under a BMS transformation. The first key step is to understand the behaviour of cut functions under a BMS transformation. Although, not within the per-

[6] This choice could be thought as switching from an element of T^4 to a generic point in M^4 in a Poincaré scenario.

spective of an application in holography, this problem has been analysed in [33] and mainly in [34] where it was provided a description for the construction of null surfaces and their singularities through the eikonal equation. The starting point is (6.1), a section of the null bundle $S^2 \times \mathbb{R}$, which satisfies the eikonal equation

$$g^{ab}\partial_a Z \partial_b Z = 0\,,$$

and which, in a more physical language, represents the family of all asymptotic plane waves. An important feature of the above equation is that, if a special solution, say $Z_0(x^a, z, \bar{z})$, is known, then all the solutions are known as well and, in particular, we can construct all the characteristic (level) surfaces of some Z given by $u = Z = const$ through the equation

$$Z(x^a, z, \bar{z}) = Z_0(x^a, z, \bar{z}) + \alpha(z, \bar{z})\,, \tag{6.2}$$

together with $\partial[Z - \alpha] = 0$ and $\bar{\partial}[Z - \alpha] = 0$. In an asymptotically flat space-time the special solution can be chosen as the cut function $u = Z(x^a, z, \bar{z})$ that we have introduced before and (6.2) describes its behaviour under the action of a supertranslation. Moreover, in this scenario, we can also introduce important variables:

$$\Lambda = \partial\left\{(1 + z\bar{z})\partial Z\right\}\,, \qquad R = (1 + z\bar{z})^2\partial\bar{\partial}Z\,,$$

representing respectively the acceleration along $(z, \bar{z})$ constant curves and the extrinsic curvature of the light cone cut; they also allow for an explicit determination of caustics through the vanishing of the following determinant:

$$D = \begin{vmatrix} \Lambda & R \\ R & \bar{\Lambda} \end{vmatrix} = 0\,. \tag{6.3}$$

Surprisingly it is more tricky to understand the effect of a Lorentz transformation on a cut function. This issue has been addressed in [35] and to some extent in [36] and [37] where it was shown that, acting with (2.3) on Z, it transforms as:

$$Z'(x'_a, z', \bar{z}') = K\left(Z(x^a, z, \bar{z})\right)\,,$$

where K is given by (2.4). If we expand in spherical harmonics and if we separate the contribution of the first four harmonics, we end up with

$$Z = x^a l_a(z, \bar{z}) + \sum_{l\geq 2} z_{lm} Y_{lm}(z, \bar{z})\,,$$

where l^a is the 4-vector $(Y_{00}, \ldots, Y_{11})$. Expanding in a similar way Z' i.e.

$$Z = x'^a l_a(z', \bar{z}') + \sum_{l\geq 2} z'_{lm} Y_{lm}(z', \bar{z}'),$$

we can see that a Lorentz transformation maps the point x^a to

$$x'^a = \Lambda^a_b x^b + \Lambda_{lm}(\Lambda^a_b) z^{lm}(x^a)\,. \tag{6.4}$$

The interpretation of the role of x'_a is rather tricky since, if we consider the Lorentz transformation in a passive sense, then Z' is simply the same characteristic function seen from a new coordinate system and, thus, it still represents the light cone with apex x^a. On the opposite, if we give an active sense to the Lorentz transformation, i.e. we interpret (6.4) as a new space-time point, we are promoting x'_a to a new bulk point and the Lorentz transformation plays the role of a diffeomorphism of the interior manifold; within this approach we pay the price that the original cut function is mapped to a different cut which is not in general the future light cone of the point x'_a. Thus this implies that a boundary Lorentz transformation (and more generally a Poincaré transformation) cannot be promoted to a bulk one and in a geometrical language this implies (see [35]) that, although at $\mathrm{Im}^{\pm}$ an asymptotically flat space-time M^4 do indeed become flat, there is no unique Minkowski space-time that can be associated to M^4 in a neighbourhood of future (or past) infinity.[7] Thus we can interpret the coordinates x^a as associating for every fixed Bondi reference frame a flat space-time with the same Im; space-time points are labeled by x^a and a minkowskian metric η^{ij} is introduced in addition to the bulk one g^{ij}.

Cut functions and holography. Since our aim is to consider the cut functions as the fundamental variables for the boundary theory, let us make some remarks on their properties from an holographic perspective:

- the function $Z(x^a, z, \bar{z})$ is a non local variable [32]. This is an important feature since we require that the theory living on Im satisfies the holographic entropy bound which cannot be reproduced by an ordinary local quantum field theory.
- if we consider a subclass of space-times "sufficiently close to Minkowski", namely asymptotically simple manifolds where vacuum Einstein equations are imposed and asymptotically flat space-times obtained from hyperboloidal initial data[8] [37], the first four coefficients of the spherical harmonics expansion of (6.1) (i.e. $k_i = \langle e_i, Z\rangle$ with $e_i = Y_{lm}(\theta, \varphi)$ $l = 0, 1$) define a canonical global coordinate system for the whole space-time. Let us emphasise that, although in the above remark we restrict the analysis to a subset of an asymptotically flat manifolds, nonetheless it represents the collection of the most physical relevant manifolds and, thus, we are not losing sensible informations performing the above choice.
- as explained more in detail in appendix B, the set of cut functions allows for a direct reconstruction of the conformal data of the bulk space-time. Thus we can look at (6.1) either only as a kinematical data for the boundary theory either as a dynamical one satisfying (B.11), (B.12), (B.13), (B.14) which are equivalent to Einstein vacuum equations.

[7] In some sense this is related to the property of the BMS group that it does not exists a unique Poincaré subgroup and the origin of this peculiarity comes exactly from the higher harmonics in the expansion in spherical harmonics of a supertranslation.

[8] It has been conjectured that this class of manifolds is a subset of asymptotically simple space-times although this has not been yet demonstrated.

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After these remarks, let us now consider the space of cut functions $\{\mathcal{C}_x\}$ (with the bulk point x^a acting as a parameter) as our configuration space and let us analyse the consequence in a BMS invariant field theory with the following modification:

$$\phi^\lambda : L^2(S^2) \to V \longrightarrow \phi^\lambda : \{\mathcal{C}_x\} \to V\,,$$

where V is a suitable target vector space. Due to the difficulties to explicitly construct (6.1) for a fixed space-time, we will first refer to Minkowski background and, after, we will make some considerations for a general bulk manifold.

Minkowski space-time. Let us start now considering the leading example of an SU(2) scalar field:

$$\phi : \{\mathcal{C}_x\} \to \mathbb{R},$$

and, for sake of simplicity, we also require that the set of null surfaces $\{C_x\}$ is constituted only by functions satisfying a suitable regularity condition. This request is not very restrictive since as mentioned before, cut functions are always smooth at least in a suitable neighbourhood of Im which implies that we can apply the formalism used in the previous section in a straightforward manner. The only subtlety comes from the necessity to introduce the equivalent of the supermomentum space in the pure BMS setting; the dual space can be constructed as the set of functions $Z^*(k^a, z, \bar{z})$ (either smooth or square integrable in the complex variables) depending on the 4-dimensional bulk momentum k^a.

Thus, repeating the calculation as in section 5, we end up with a 2-point function as in (5.4) whose argument now depends on Z^*_k (or rather on its projections along the spherical harmonics):

$$\left[\eta^{\mu\nu}k_\mu k_\nu - m^2 + \sum_{i>4}\frac{1}{2\zeta_i}[-k_i^2 + k_i D_{e_i}]\right]\hat{D}(Z^*_k) = i\,,$$

with $k_\alpha = \langle Z^*(k^a, z, \bar{z}), e_\alpha\rangle$. As mentioned before, this is a functional differential equation and, as far as we know, there is no way to solve it analytically. Nonetheless, in the peculiar case of a flat background, the cut-function has been constructed and it has a simple form [32, 34]:

$$Z(x_a, z, \bar{z}) = x_a l^a(z, \bar{z}) \tag{6.5}$$

$$l^a(z, \bar{z}) = \frac{1}{\sqrt{2}(1+z\bar{z})}\left((1+z\bar{z}), -(z+\bar{z}), i(z-\bar{z}), (1-z\bar{z})\right), \tag{6.6}$$

where x^a is the canonical minkowskian global coordinate system. Moreover, in order also to understand the simplicity of (6.5), we have to take into account that, in a flat manifold, the BMS group can be univocally reduced to the Poincaré group which is reflected in the above formula through the absence of a pure supertranslational part. Thus, switching to the (super)momentum frame changing the x_a-variable with k_a, we can substitute $Z^* = k_a l^a(z, \bar{z})$ in (5.4); since each $k_i = 0$ and since we can interpret the coefficients k^a of the first four harmonics as a minkowskian coordinate frame, we end up exactly with the Poincaré propagator:

$$D(k) = \frac{i}{\eta^{\mu\nu}k_\mu k_\nu + m^2}\,.$$

General background. In a general background the analysis of the boundary theory and in particular of (5.4) is rather more complicated even after the introduction of cut functions. If we consider a four dimensional asymptotically flat manifold $(M^4, g^{\mu\nu})$, the first step would be the construction of possible cut functions in the chosen space-time and this operation can be technically difficult. At present, $Z(x^a, z, \bar{z})$ has been successfully analysed only in few specific examples namely for the Schwartzschild and the Kerr metric [39, 40]. In order to better clarify this difficulty and in order to leave the paper self-consistent, we briefly review the construction of the cut function for the Schwartzschild case; in a Bondi reference frame (u, r, θ, φ), where $u = t - r - 2m\log(r - 2m)$, the metric is

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,,$$

which can be conformally transformed by $\Omega = r^{-1} = l$ in

$$d\hat{s}^2 = \Omega^2 ds^2 = 4(l^2 - \sqrt{8}ml^3)du^2 - 4dudl - (d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{6.7}$$

In order to simplify the calculation, it is worth to notice that, due to spherical symmetry, we can first investigate the light cone equation on the equatorial plane $\theta = \frac{\pi}{2}$ and then we can generate all the possible solutions by a rigid rotation. Thus, starting form (6.7) and switching from (θ, φ) to complex coordinates $(z, \bar{z})$ via stereographic projection, the light cone equation is

$$2(l^2 - \sqrt{8}ml^3)\dot{u} - \dot{l} = 1\,, \qquad \ddot{u} + 2\left(l - \sqrt{18}ml^2\right)\dot{u}^2 = 0\,, \tag{6.8}$$

$$\ddot{\varphi} = 0\,, \qquad \dot{\varphi} = b\,, \qquad (l^2 - \sqrt{8}ml^3)\dot{u}^2 - \dot{u}\dot{l} = \frac{b^2}{4}\,, \tag{6.9}$$

where b is a constant. The above equations can be written in the following form:

$$\dot{u} = \frac{1 + \dot{l}}{2(l^2 - \sqrt{8}ml^3)}\,, \qquad \dot{l} = \pm(\sqrt{8}mb^2l^3 - b^2l^2 + 1)^{\frac{1}{2}} = \pm\sqrt{A}\,, \tag{6.10}$$

$$ds = \pm\frac{dl}{\sqrt{A}}\,, \qquad d\varphi = \pm\frac{b}{\sqrt{A}}dl\,. \tag{6.11}$$

Keeping fixed the light cone apex $x_a = (u_0, l_0, z_0, \bar{z}_0)$, we can realize from (6.10) that the possible solutions to the above differential equations are divided into two sets depending on the sign of $\dot{l}$. If we consider $\dot{l} < 0$, we can integrate (6.10), (6.11) and, after a lengthy calculation, the final result is

$$\begin{aligned} u &= u_0 - \frac{1}{2}\int_{l_0}^{0} \frac{b^2}{\sqrt{A}}dl' + \frac{1}{2}\int_{l_0}^{0} \frac{b^2}{\sqrt{A}+1}dl'\,, \\ \varphi - \varphi_0 &= -\int_{l_0}^{0} \frac{b}{\sqrt{A}}dl'\,. \end{aligned} \tag{6.12}$$

In order to obtain the cut function, we have to perform the rotation along the θ-direction and this can be achieved substituting the latter equation with

$$\varphi - \varphi_0 = \arccos(1 - l_a(z, \bar{z})l^a(z_0, \bar{z}_0)) = -\int_{l_0}^{0} \frac{bdl'}{\sqrt{A}}\,, \tag{6.13}$$

where l_a is given by (6.6). Since we cannot solve explicitly the above elliptic integrals, it is impossible to give an explicit expression for $u = Z(x^a, z, \bar{z})$ and the latter can be written only parametrically via (6.12) and (6.13).

Although a similar solution can be calculated for $\dot{l} > 0$, this would not improve our understanding of the construction and of the properties of the cut function in the Schwartzschild background. Instead, in the spirit of finding an holographic correspondence in an asymptotically flat space-time, it is important to stress that the above procedure confirms the peculiar nature of the flat scenario[9] where it is easier to study an holographic correspondence due to the presence of an explicit analytic cut function. Nonetheless, although in the Schwartzschild case, the boundary data can be written only in a parametric form, it is worth to notice that we still deal with a cut function without any supertranslational component; thus, despite the presence of elliptic integrals, it is still conceivable, in a future research, to look for a solution to the propagator differential equation (5.4) in this background. As a final remark, let us emphasize that the absence of any spherical harmonic with $l > 1$ should not surprise since it is known that the BMS group could be univocally reduced to the Poincaré group by means of geometric arguments in any stationary space-time and the equations (6.12) and (6.13) simply reflect this peculiar property.

A second step, in order to study holography in an asymptotically flat manifold via null surface formulation of general relativity, consists in the study of the differential equation for the propagator in order to construct an explicit solution of (5.4); although, as mentioned before several times, we lack a technique to complete this task whenever the BMS group cannot be reduced in a unique way to the Poincaré group as in Minkowski space, we can nonetheless draw some interesting conclusions: first of all, with the introduction of cut functions, we can interpret the coefficients k_μ as global coordinates in the bulk manifold and, since the pure supertranslations act as evolution directions, we can interpret the differential equation as connecting different Minkowski space-times. We can also be tempted to conjecture that the BMS field theory, together with the cut functions, allows for the reconstruction of the bulk data in a neighbourhood of future (past) infinity. A natural question arising immediately is whether it is conceivable to extend the reconstruction of bulk data far beyond those points where the metric can be approximated to a good degree to $\eta^{\mu\nu}$. Although an answer to this question might be premature, we wish nonetheless to conjecture a possible solution which consists in the following picture: until now, the set of cut functions acted as a pure kinematical data. Nonetheless, as explained in appendix B, we can think to $Z(x^a, z, \bar{z})$ as a dynamical data allowing us to reconstruct univocally the (conformal) bulk metric and satisfying a certain set of equations equivalent to Einstein's vacuum equations (B.11), (B.12), (B.13), (B.14) for general relativity. Switching to this point of view, one is entitled to substitute in the BMS action the flat metric with a generic metric which is no more an independent data, but it is a now dependent on the cut functions i.e.

$$S = \int_{L^2(S^2)} d\mu(Z_x)\phi(Z_x)[g^{\mu\nu}(Z_x)D_{e_\mu}D_{e_\nu}\phi(Z_x) - m^2\phi(Z_x)] + \sum_{i=4}^{\infty}\gamma_i(Z_x)D_{e_i}\phi(Z_x)\,. \quad (6.14)$$

[9]Let us notice that, in the limit of a vanishing Schwartzschild mass $m \to 0$, (6.12) and (6.13) converge to the Minkowski cut function (6.5).

Let us emphasise that this modification does not change any result at a classical level with the due exception of the mass relation:

$$m^2 = \eta^{\mu\nu} p_\mu p_\nu \to m^2 = g^{\mu\nu}(Z_x) p_\mu p_\nu \,.$$

Thus we believe that it could be interesting to study if, (6.14) together with the equations for the cut functions (B.11), (B.12), (B.13), (B.14), allows for a complete holographic reconstruction of bulk data.

7. Conclusions

In this paper we have studied some aspects of a quantum BMS field theory and the relations with an holographic correspondence in asymptotically flat space-times. In particular, through the leading example of a massive BMS scalar field we have succeeded in constructing an action for this particle and we have studied its partition and 2-point correlation function. Surprisingly the latter turned to be quite different from its counterpart in an ordinary Poincaré invariant quantum field theory since it is a differential equation. Far from being able to analytically solve it, we have nonetheless draw some conclusions namely the pure supertranslational components of the 2-point function, which are sterile at a classical level, act as evolution directions. Thus we have been able to interpret (5.4) as an equation connecting the value of the propagator from one Poincaré subgroup of the BMS to another; from one side this is not surprisingly since there is not a unique way to construct a Minkowski background in a neighbourhood of Im, namely one for each element of $L^2(S^2)/T^4$, whereas from the other side it seems interesting to analyse such result from the perspective of Ashtekar asymptotic quantisation program [30] where a key ingredient is played by infrared sectors each of them labeled by a pure supertranslational element. Thus it seems worth on its own for a future work to study BMS field theory and its characteristics despite its relation with holography. On this side instead, the construction of BMS actions and correlation functions has emphasised that BMS invariance plays a necessary but not a sufficient role to completely understand a possible bulk to boundary correspondence. In particular the role of supertranslation $L^2(S^2)$ as main kinematical data seems lacking insights on how to fully understand the boundary theory in the same way as T^4 does not allow to plainly understand a Poincaré invariant theory in an (asymptotically) flat background.

Thus we have introduced a new ingredient in our analysis: cut functions i.e. maps $Z : M^4 \times S^2 \to \mathbb{R}$. As explained in the previous section and in appendix B, in the null surface formulation of gravity they allow for a full reconstruction of the metric and of Einstein's equations and, more important from our point of view, they can be interpreted as boundary data. Using these functions as the main variables and keeping in mind that the first four coefficients of the spherical harmonics expansion of (6.1) define a global pseudo minkowskian coordinate system, we have been able to relate the BMS correlation function with its bulk counterpart in a flat background.

In order to clarify the overall picture and, since in this paper we have always referred to a boundary scalar field, we wish to briefly switch the perspective commenting on how the reconstruction of a bulk scalar field works in the language of null surface formulation. If we consider a Minkowski background, the classical dynamic of a free bulk scalar fields is completely equivalent to its BMS counterpart. As we have shown in the previous sections, the pure supertranslational components are sterile and the BMS-Klein-Gordon equation is defined on the mass hyperboloid as in a Poincaré invariant theory. The big difference instead lies at a quantum level; since the quantum BMS field theory does not distinguish between different asymptotically flat space-times, the data of a bulk scalar field can be reconstructed only if we add to the boundary BMS scalar field a specific choice of a cut function $Z(x^a, z, \bar{z})$. As we have shown in section 6, the latter data allows for a selection in (5.4) of a specific solution of the differential equation and the correct formula for the 2-point correlation function for a free scalar field living in a flat manifold is retrieved. In a more generic setting, instead, the difficulties we face in order to solve the differential equation has not allowed us to draw a similar conclusion although it seems rather natural to conjecture that a similar procedure holds for every asymptotically flat space-time and for any free field. To be more precise, we need to emphasise that Minkowski is in a lot of ways a peculiar example since we can reconstruct the whole bulk with a single square-integrable cut function. On the opposite in a generic manifold, the function $Z(x^a, z, \bar{z})$ is neither single valued neither smooth except for a suitable neighbourhood of Im^+. Thus, in this scenario, we do not expect to reconstruct the whole bulk/boundary correspondence from a single cut function; furthermore, in future analysis, we will need to overcome two possible difficulties: the first is rather technical and it is related to the fact that light cones develop caustics i.e. points where the description of the space-time through the eikonal function fails and these points can be characterised by the divergence of (6.3). A second problem comes from the flat metric which is intrinsic in the BMS action; this is a natural ingredient for a theory living on $\mathrm{Im}^\pm$ where the metric is indeed $\eta^{\mu\nu}$ but, it seems to us that, if we wish to reconstruct the data of fields living on a curved bulk background, we lack the information from the metric itself. A possible way to overcome this problem has been suggested at the end of the previous section substituting $\eta^{\mu\nu}$ with a generic metric $g^{\mu\nu}(Z)$ which is dependent on the cut function itself. This trick would in principle allow us to solve the above mentioned problem but calculations for the propagator appear now rather more difficult.

As a final remark, we wish to comment on the construction of an S-matrix within the setting we have considered. As it has been pointed out in [7], the overall picture, which emerges from a BMS approach to holography, is rather similar to 't Hooft analysis in the context of black holes (see [45] and [46] for a recent review and developments). Without entering into details, in this latter scenario, an S-matrix ansatz is assumed and the description is given in a first quantized set up; the holographic fields live on the future and on the past horizon and they depend only upon angular coordinates. If we consider two such fields, namely $u(\theta, \varphi)$ and $v(\theta, \varphi)$, the high degree of non locality of this approach is fully encoded in the operator algebra

$$[u(\theta, \varphi), v(\theta', \varphi')] = if(\theta - \theta', \varphi - \varphi'),$$

where f is the Green function of the laplacian operator acting on the angular horizon coordinates. If we take into account that non locality is a fundamental request of an holographic theory and if we consider the peculiar dependance upon the angular variables, the similarity with the data of a BMS field theory developed in this paper are rather surprising; thus the overall picture which emerges from our analysis supports the conjecture that it is possible to construct an S-matrix in an asymptotically flat space-time with "in" and "out" states living respectively on Im^+ and Im^- with fields carrying BMS labels. Furthermore we believe that this idea is not antithetical to the philosophy of this paper where we have focused our attention on developing a quantum BMS field theory living on Im^+ or on Im^- and where we have looked for an implementation of the holographic correspondence through an analysis of the correlation functions. On the contrary, the introduction of the null surface approach as a tool to compare bulk and boundary data has shown that the angular dependence is a necessary feature of a BMS field as in 't Hooft approach. Nonetheless a direct construction of such a mapping is still a big task far from being complete; as pointed in [8], one of the main difficulties in this direction lies in the existence of non trivial IR sectors which are labeled by pure supertranslation. As suggested at the end of section 5 and more in detail in section 6, this peculiar behaviour is encoded in the boundary data by means of the specific dependence of BMS fields upon cut functions with a non vanishing pure supranslational term; thus we believe that this specific characteristic of the boundary data should be one of the key aspects in the construction of the S-matrix but a solution of this task is still not available and this topic is left for future investigations.

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A. Elements of white noise as an infinite dimensional calculus

The aim of this appendix is to give some knowledge of infinite dimensional calculus and white noise analysis in order to keep the paper self-contained. We will concentrate mainly on definitions and applications that have been useful throughout the paper and we leave the details of demonstrations and the development of deeper applications to foundational books [23, 24].

Let us consider a real separable Hilbert space $\mathcal{H}$ with norm $||,||$ and let us consider an operator A in $\mathcal{H}$ such that there exists an orthonormal basis $\{\zeta_j\}$ satisfying the following conditions:

1. $A\zeta_j = \lambda_j \zeta_j\,, \qquad \forall j$
2. $1 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_j \leq \cdots\,,$
3. there exists $\alpha \in \mathbb{R}_+$ such that $\sum \lambda_j^{-\alpha} < \infty$.

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If, as an example, we consider the Hilbert space $L^2(S^2)$ and the canonical base of spherical harmonics any operator $J^2 + \epsilon$, where $\epsilon > 1$ and where J is the angular momentum, plays the role of A.

The white noise space associated to $(\mathcal{H}, A)$ can be constructed in the following way: for each positive integer p define the element $|\zeta|_p = ||A^p\zeta||$ and the set $\mathcal{E}_p = \{\zeta \in \mathcal{H}; |\zeta|_p < \infty\}$. Let us call the projective limit of $\{\mathcal{E}_p \;\; p > 0\}$ as $\mathcal{E}$ and $\mathcal{E}'$ its dual. This allows us to define the Gelfand triplet

$$\mathcal{E} \subset \mathcal{E}_p \subset \mathcal{H} \subset \mathcal{E}'_p \subset \mathcal{E}' , \qquad p \geq 0 .$$

Through the Minlos theorem [27] we can introduce the unique probability measure $\mu \in \mathcal{E}'$ and the complex Hilbert space of square-integrable functions respect to this measure i.e. $\mathcal{H}' = L^2(\mathcal{H}, \mu)$. From now on this will be our reference space and we will introduce operators acting on $\mathcal{H}'$. The first step is to define a Gelfand triplet also for this Hilbert space; this task can be achieved through the Ito-Wiener lemma which allows us to uniquely express each element $\phi \in \mathcal{H}'$ as

$$\phi = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in \mathcal{H}_c^{\otimes n}$$

where $\mathcal{H}_c^{\otimes n}$ is the complexification of the symmetric tensor product of $\mathcal{H}$ and where I_n is the multiple Wiener-integral (see section 3 in [24] for an analysis). Let us now introduce the operator $\Gamma(A)$ such that

$$\Gamma(A)\phi = \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n) .$$

We can now repeat the construction of a Gelfand triplet defining for each $p > 0$ the norm

$$||\phi||_p = ||\Gamma(A)^p \phi|| ,$$

and, consequently,

$$\mathcal{T}_p = \left\{ \phi \in L^2(\mathcal{H}) \qquad ||\phi||_p < \infty \right\} .$$

Introducing the projective limit of $\{\mathcal{T}_p \;\; p > 0\}$ as $\mathcal{T}$ and its dual space $\mathcal{T}^*$, we have the Gelfand triplet associated to $\mathcal{H}'$

$$\mathcal{T} \subset \mathcal{T}_p \subset \mathcal{H}' \subset \mathcal{T}_p^* \subset \mathcal{T}^* .$$

Let us now introduce some suitable operations on the above varieties. In particular let us define the so-called *S-transform* of an element $\Phi \in \mathcal{T}^*$ as the function

$$S\Phi(x) = \langle\langle \Phi, : e^{\langle \cdot, x \rangle} : \rangle\rangle , \tag{A.1}$$

where $\langle\langle , \rangle\rangle$ is the internal product on $\mathcal{H}' = L^2(\mathcal{H}, d\mu)$ (in order to avoid confusions with the norm $||, ||$ in $\mathcal{H}$) and where

$$: e^{\langle x, h \rangle} := e^{\langle x, h \rangle - \frac{1}{2}||h||^2} .$$

A second transformation we can introduce is the $\mathcal{T}$*-transform* which assigns to each function $\Phi \in \mathcal{T}^*$ the function

$$\mathcal{T}\Phi(x) = \langle\langle \Phi, e^{i\langle \cdot, x\rangle}\rangle\rangle\,. \tag{A.2}$$

Let us now introduce continuous linear operators acting on the space of test-functions; in particular for any $\phi \in \mathcal{T}$ we can define the *Gateaux derivative* or the directional derivative along the direction $y \in \mathcal{E}'$ as

$$D_y\phi(x) = \lim_{\epsilon\to 0} \frac{\phi(x+\epsilon y) - \phi(x)}{\epsilon}\,. \tag{A.3}$$

Let us stress that, given $\eta \in \mathcal{E}$, the operator D_η admits a unique extension by continuity to a continuous linear operator $\tilde{D}_\eta$ on $\mathcal{T}^*$.

A second fundamental operator which has to be defined on $\mathcal{T}$ is the *multiplication operator* Q_η ($\eta \in \mathcal{E}$):

$$Q_\eta\phi(x) = \langle x, \eta\rangle\phi(x)\,. \tag{A.4}$$

As for the Gateaux derivative, the multiplication operator admits a unique extension by continuity to a continuous linear operator $\tilde{Q}_\eta$ on $\mathcal{T}^*$. A further important concept for the theory of operators acting on Hilbert spaces is the adjunction i.e., for any operator $J \in \mathcal{T}$, we define the adjoint operator $J^* \in \mathcal{T}^*$ as

$$\langle\langle J^*\Phi, \phi\rangle\rangle = \langle\langle \Phi, J\phi\rangle\rangle\,, \qquad \Phi \in \mathcal{T}^*, \phi \in \mathcal{T}\,. \tag{A.5}$$

Moreover, if J is a continuous linear operator, then this property holds also for the adjoint operator.

We stress to the reader the difference between the operators D^*_η and $\tilde{D}_y$ albeit both are defined on $\mathcal{T}^*$. This difference can be further emphasised by the following commutation relation:[10]

$$\left[\tilde{D}_\eta, D^*_y\right] = \langle y, \eta\rangle I \qquad \text{on } \mathcal{T}^* \;\forall \eta \in \mathcal{E}, y \in \mathcal{E}' \tag{A.6}$$

$$\left[D_y, D^*_\eta\right] = \langle y, \eta\rangle I \qquad \text{on } \mathcal{T} \;\forall \eta \in \mathcal{E}, y \in \mathcal{E}'\,. \tag{A.7}$$

A useful set of identities that holds and relates the multiplication and the derivative operators is the following:

$$Q_\eta = D_\eta + D^*_\eta\,, \tag{A.8}$$

$$\tilde{Q}_\eta = \tilde{D}_\eta + D^*_\eta\,. \tag{A.9}$$

An important consequence of (A.8) is the definition of integration by parts within the realm of gaussian measures i.e., for any $\phi, \psi \in \mathcal{T}$,

$$\int_{\mathcal{E}'} (D_\eta\phi)\,\psi(x)d\mu(x) = -\int_{\mathcal{E}'} \phi(x)\left[(D_\eta\psi)(x) - \langle x, \eta\rangle\psi(x)\right] d\mu(x)\,.$$

[10] Here we only refer to the non vanishing commutation relations.

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We are now in position to introduce the concept of *Fourier transform* $\hat{\Phi}$ of a generalised function $\Phi \in \mathcal{T}^*$ as the unique element whose S-transform is

$$S\hat{\Phi}(x) = (-ix)(S\Phi)e^{-\frac{1}{2}\langle x,x\rangle}\,, \qquad \forall x \in \mathcal{E}_c$$

where the subscript stands for the complexification of $\mathcal{E}$. At a level of transformations we can relate the Fourier transform $\mathcal{F}$ to the other transforms as

$$\mathcal{F} = \mathcal{T}^{-1}S\,. \tag{A.10}$$

Although the above definition is highly non trivial, $\mathcal{F}$ shares several common properties with its finite-dimensional counterpart. In particular, when it is applied to the differential and multiplication operators, the following identities hold:

$$\mathcal{F}\tilde{D}_\eta = i\tilde{Q}_\eta\mathcal{F}\,, \tag{A.11}$$

$$\mathcal{F}D_x^* = -iD_x^*\mathcal{F}\,, \tag{A.12}$$

$$\mathcal{F}\tilde{Q}_\eta = i\tilde{D}_\eta\mathcal{F}\,. \tag{A.13}$$

A last important remark is related to a reiterate application of the Fourier transform; in order to correctly address this issue, let us introduce the *reflection map* $\rho : \mathcal{T}^* \to \mathcal{T}^*$ such that (see [23])

$$S(\rho\Phi)(x) = S\Phi(-x)\,.$$

Thus the following identity holds:

$$\mathcal{F} = \rho S^{-1}\mathcal{T}\,,$$

and, from (A.10),

$$\mathcal{F}^2 = \rho \qquad \text{and} \qquad \mathcal{F}^4 = Id\,.$$

Let us stress that, with the due exception of this appendix, throughout the paper the domain of definition of all the functions is implicit and it is never explicitly stated.

B. Null surface formulation of general relativity

In this appendix we review the main developments and techniques that lead to an equivalent formulation of general relativity through characteristic surfaces since we believe that this approach to the description of the dynamic of the gravitational field plays a fundamental role in the quest to find an holographic description in asymptotically flat space-times.

Let us consider an asymptotically flat space-time M with boundary $\mathrm{Im}^\pm$ and a given metric g_{ab}; if we consider a point $x_a \in M$, we can construct the light cone with apex x_a through the null geodesic equation $L(x_a, x'_a) = 0$ where x'_a is a generic point on the cone. If we send x'_a to the boundary Im^+, we can introduce a Bondi coordinate frame $(u, r = 0, z, \bar{z})$ and we can rewrite the light cone equation as [32]:

$$L(x_a, u, z, \bar{z}) = 0\,, \tag{B.1}$$

which can be inverted in the variable u as

$$u = Z(x_a, z, \bar{z})\,, \tag{B.2}$$

which will be the fundamental equation in the null surface description of general relativity. It is imperative to stress that, since (B.2) is the inverse of (B.1), a priori it is not globally differentiable or single valued. Nonetheless, at least locally (i.e. in a suitable open set of $x'_a = (u, z, \bar{z})$), the function Z is either differentiable either single valued. Moreover we can draw the following conclusions on the Z-function:

- keeping fixed the point x_a in (B.2), this equation describes the so called *light-cone cut* i.e. the intersection $\mathcal{C}_{x_a}$ between the light cone N_{x_a} with apex x_a and the boundary Im^+ or, more analytically, Z is a map from the S^2-set of light like directions starting from x_a into a suitable two dimensional surface embedded in Im^+.
- keeping fixed instead the boundary point $x'_a = (u, z, \bar{z})$, (B.2) describes the past light-cone of x'_a.
- the function $Z(x_a, z, \bar{z})$ is highly non local.
- at a geometric level, $Z(x_a, z, \bar{z})$ is a section of the null bundle $(M^4 \times S^2, M^4, \pi)$ where π is the trivial projection and where

  $$g^{ab} Z_{,a} Z_{,b} = 0,$$

  i.e. Z is a characteristic surface.

Let us stress that the light cone cuts are not regular surfaces except in some peculiar case as Minkowski space-time. In general, these surfaces can develop caustics or self-interactions but nonetheless $\mathcal{C}_{x_a}$ will be always homotopically equivalent to S^2 [38]. In the language of fiber bundles this is equivalent to state that the function Z is not a global section but it is only local and that the light cone cuts can be described as suitable projections of smooth two-dimensional surfaces embedded in a larger space i.e. the cotangent bundle $T^*(\mathrm{Im}^+)$.

Since the key idea is to consider (B.2) as the fundamental variable, a natural question which arises is if it is possible to construct a metric $g_{ab}(x)$ (which is the fundamental field in general relativity) starting from an arbitrary function $Z(x_a, z, \bar{z})$ which also implies that the cuts $\mathcal{C}_{x_a}$ belong only to a certain class of non local function capable of producing a local field in the bulk M^4. The question has been dealt with and it has been answered in [32] where the authors impose suitable conditions both on the metric components and on the set of functions $Z(x_a, z, \bar{z})$ in order to correctly associate to each Z a non vanishing symmetric tensor field $q_{ab}(x)$ such that

$$q^{ab}(x) Z_{,a} Z_{,b} = 0\,.$$

Without entering the details of the derivation, let us only review the main results; let us introduce the *edth* derivative operator $\eth$ (see [41] for a definition) and the tetrad basis $\Theta^i_a = (Z_{,a}, \eth Z_{,a}, \bar{\eth} Z_{,a}, \eth\bar{\eth} Z_{,a})$ in such a way that the metric components can be written as

$$q^{ij} = q^{ab} \Theta^i_a \Theta^j_b\,.$$

Applying recursively the derivatives $\eth$ and $\bar\eth$, we end up with the following constraints on q^{ij}:

$$q^{00} = q^{0+} = q^{0-} = 0\,, \tag{B.3}$$

$$\frac{q^{+-}}{q^{01}} = -1 \qquad \frac{q^{++}}{q^{01}} = -\Lambda_1 \qquad \frac{q^{--}}{q^{01}} = -\bar\Lambda_1\,, \tag{B.4}$$

$$\frac{q^{+1}}{q^{01}} = -\frac{1}{2}(\bar\eth\Lambda_1 + \Lambda_1\bar W) \qquad \frac{q^{-1}}{q^{01}} = -\frac{1}{2}(\eth\bar\Lambda_1 + \bar\Lambda_1 W,)\,, \tag{B.5}$$

$$\eth\ln q^{01} = W \qquad \bar\eth\ln q^{01} = \bar W\,, \tag{B.6}$$

$$\frac{q^{11}}{q^{01}} = -2\epsilon(\Lambda_0,\Lambda_1,\Lambda_-)\,, \tag{B.7}$$

where ϵ is a real function (see appendix A in [32] for an explicit expression), $\Lambda(x,z,\bar z) = \eth^2 Z$, $\Lambda_{,a} = \Lambda_i\Theta^i_a$ and $W = \frac{\partial\Lambda_1 - 2\Lambda_-}{\Lambda_1} + \partial\ln P$ with $P = (1-\Lambda_1\bar\Lambda_1)^{-1}$. The cut function $Z(x_a,z,\bar z)$ or, more properly, its second derivative has to satisfy the following equations:

$$\Lambda_+ = W - \frac{1}{2}\left(\Lambda_1\bar W + \bar\eth\Lambda_1 + \eth\ln P\right) \quad \text{and c.c.}\,, \tag{B.8}$$

$$\eth\epsilon + \epsilon(W-2\Pi_1) + \left(1-\frac{\Pi_+}{2}\right)(\bar\eth\Lambda_1+\Lambda_1\bar W) - \frac{1}{2}(\eth\bar\Lambda_1+\bar\Lambda_1 W)\Pi_- + \Pi_0 = 0 \quad \text{and c.c.}\,, \tag{B.9}$$

where Π is a function of Z and its derivatives (still see appendix A in [32] for a definition). Moreover if we want to define a lorentzian metric, we have to impose a further condition:

$$\det(q_{ab})\,\langle 0 \Longrightarrow \det(q_{ij}) = (q^{01})^4 P > 0\,, \tag{B.10}$$

or equivalently $P > 0 \Longleftrightarrow |\Lambda_1| < 1$.

Thus the set of functions satisfying the above conditions allow us to write a lorentzian metric and, consequently, the connection and curvature tensor as functionals only of the cut equation. Nonetheless this condition is not sufficient since this does not grant us that the metric satisfies Einstein equations. In order to impose also this requirement, one has to realize [42, 43] that, in order to reformulate general relativity in terms of equations for families of surfaces, one also needs to introduce a second function $\Omega(x^a,z,\bar z)$ that acts as a conformal factor turning the conformal metrics constructed from the characteristic surfaces $u = Z(x^a,z,\bar z)$ into an Einstein metric. Thus the vacuum Einstein equations become (see as an example [44])

$$D^2\Omega = Q(\Lambda)\Omega\,, \tag{B.11}$$

where

$$Q = -\frac{1}{4q}(D\bar\Lambda_1 D\Lambda_1) - \frac{3}{8q^2}(Dq)^2 + \frac{1}{4q}D^2 q\,, \quad q = 1-\Lambda_1\bar\Lambda_1 \quad \text{and} D = \frac{\partial}{\partial(\eth\bar\eth Z)}\,.$$

Moreover both Λ and Ω have to satisfy the following consistency equations:

$$\eth\Omega = W(\Lambda_{,1})\Omega\,, \tag{B.12}$$

$$\eth\Lambda_{,1} - 2\Lambda_{,-} = (W + \eth(\ln q))\Lambda_{,1}\,, \tag{B.13}$$

$$\eth^2\Lambda = \bar\eth^2\Lambda\,. \tag{B.14}$$

At this stage we need to emphasise that, although the above set of differential equations is rather difficult to solve in a general scenario, it can be simplified for regular asymptotically flat space-times. In particular, as shown in details in [36], if we introduce the asymptotic Bondi shear $\sigma(u,z,\bar{z})$, (B.12), (B.13), (B.14) can be substituted by the equation:

$$\eth^2\bar{\eth}^2 Z = \bar{\eth}^2\sigma_R + \eth^2\sigma_R + N[Z,\Omega]\,, \tag{B.15}$$

where N is a complicated function of Λ and its derivatives (see [36] for the explicit expression) and σ_R represents the freely chosen Bondi shear with the $u-$variable substituted by $Z(x_a, z, \bar{z})$.

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